

**MARKING SCHEME AND MODEL SOLUTIONS****Prep4Uni | Semester 1 Mid-Term Test | Calculus: Derivatives | 60 marks | 90 minutes****Marking conventions**

M = method mark, **A** = accuracy mark (awarded only if the related M mark is earned), **B** = independent mark. Follow-through (ft) marks are awarded when a correct method is applied to an earlier incorrect value. Any correct alternative method earns full marks. Where a question demands the limit definition, marks are not awarded for answers found using differentiation rules alone.

Test blueprint (all questions are original; the exercise numbers show the textbook practice they parallel)

Q	Topic	Marks	Related Thomas' Calculus exercises
1	Derivative at a point from the limit definition; tangent line	5	3.1: Ex. 11-22
2	Tangent line of given slope	3	3.1: Ex. 27, 28
3	Average and instantaneous rate of change	5	3.1: Ex. 29-32
4	Interpreting derivative values from a graph	4	3.1: Ex. 23, 24
5	Vertical tangent lines	3	3.1: Ex. 39-42
6	Alternative formula for the derivative	6	3.2: Ex. 23-26
7	One-sided derivatives	4	3.2: Ex. 37-44
8	Differentiability of a piecewise function	4	3.3: Ex. 57, 58
9	Graph of f and its derivative (step function)	6	3.2: Ex. 31, 32
10	First and second derivatives (power rule)	4	3.3: Ex. 1-12
11	Product and Quotient Rules	4	3.3: Ex. 13-22
12	Derivatives from given values of u , v	3	3.3: Ex. 39, 40
13	Tangent and normal lines; horizontal tangents	5	3.3: Ex. 41, 42
14	Tangent lines through an outside point	4	3.3: Ex. 50

Section totals: Section A (Q1-Q5) 20 marks; Section B (Q6-Q9) 20 marks; Section C (Q10-Q14) 20 marks. **Total 60.**



SECTION A: Tangent Lines and the Derivative at a Point

[20 marks]

Question 1 $f(x) = 4/(x + 1)$ at $P(1, 2)$		[5]
(a)	$f(1) = 2, \quad f(1+h) = \frac{4}{2+h}$ $f'(1) = \lim_{h \rightarrow 0} \frac{\frac{4}{2+h} - 2}{h} = \lim_{h \rightarrow 0} \frac{4 - 2(2+h)}{h(2+h)}$ $= \lim_{h \rightarrow 0} \frac{-2h}{h(2+h)} = \lim_{h \rightarrow 0} \frac{-2}{2+h} = -1$	M1 correct difference quotient M1 common denominator, cancels h A1 slope = -1
(b)	$y - 2 = -1(x - 1) \Rightarrow y = -x + 3 \quad (x + y = 3)$	M1 point-slope form with slope from (a) A1 correct equation
Question 2 Tangent to $y = \sqrt{x}$ with slope 1/3		[3]
	$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ $\frac{1}{2\sqrt{x}} = \frac{1}{3} \Rightarrow \sqrt{x} = \frac{3}{2} \Rightarrow x = \frac{9}{4}, \quad y = \frac{3}{2}$ $y - \frac{3}{2} = \frac{1}{3}\left(x - \frac{9}{4}\right) \Rightarrow y = \frac{x}{3} + \frac{3}{4} \quad (12y = 4x + 9)$	B1 derivative (definition or power rule accepted) B1 point of tangency (9/4, 3/2) A1 equation
Question 3 Stone dropped from an 80 m cliff		[5]
(a)	$h(1) = 75.1, \quad h(4) = 80 - 78.4 = 1.6$ $\frac{h(4) - h(1)}{4 - 1} = \frac{1.6 - 75.1}{3} = -24.5 \text{ m/s}$	B1 both heights A1 -24.5 (sign and value)
(b)	$h(2) = 60.4, \quad h(2+k) = 80 - 4.9(4 + 4k + k^2) = 60.4 - 19.6k - 4.9k^2$ $\frac{h(2+k) - h(2)}{k} = \frac{-19.6k - 4.9k^2}{k} = -19.6 - 4.9k$ $\lim_{k \rightarrow 0} (-19.6 - 4.9k) = -19.6$ Instantaneous velocity at $t = 2$ s is -19.6 m/s. The negative sign means the height is decreasing; the stone is moving downward at 19.6 m/s.	M1 correct difference quotient, expanded A1 limit -19.6 B1 units m/s and meaning of the sign



Question 4 Water depth $W(t)$: reading the graph [4]		
(a)	The depth is increasing on about $0 \leq t < 10$ hours. On that interval the tangent lines slope upward, so $W'(t) > 0$.	B1 interval (accept 9.5 to 10.5 h) B1 W' positive
(b)	Maximum depth at $t \approx 10$ hours. The tangent is horizontal there, so $W' = 0$.	B1 both time and $W' = 0$
(c)	$W'(6)$ is the instantaneous rate of change of the depth 6 hours after the storm begins. Units: metres per hour (m/h) . <i>Teacher note: the graph was drawn from $W(t) = 1 + 0.8t - 0.04t^2$, so $W'(6) = 0.32$ m/h. This value is not required.</i>	B1 meaning and units both needed

Question 5 Vertical tangent lines at the origin [3]		
(a)	$\lim_{h \rightarrow 0} \frac{h^{1/5} - 0}{h} = \lim_{h \rightarrow 0} \frac{1}{h^{4/5}} = +\infty$ Since $h^{4/5}$ is positive on both sides of 0, the limit is $+\infty$ from both sides: f has a vertical tangent line (the line $x = 0$).	B1 limit $+\infty$ with conclusion
(b)	$\lim_{h \rightarrow 0} \frac{h^{4/5} - 0}{h} = \lim_{h \rightarrow 0} \frac{1}{h^{1/5}}$ As $h \rightarrow 0^+$ this tends to $+\infty$; as $h \rightarrow 0^-$ it tends to $-\infty$. The limit does not exist, so g has no vertical tangent at the origin (the graph has a cusp).	B1 the two one-sided behaviours B1 conclusion with reason

SECTION B: The Derivative as a Function

[20 marks]

Question 6 $f(x) = 2/(x + 3)$ [6]		
(a)	$f(z) - f(x) = \frac{2}{z+3} - \frac{2}{x+3} = \frac{2(x+3) - 2(z+3)}{(z+3)(x+3)}$ $= \frac{2(x-z)}{(z+3)(x+3)}$ $\frac{f(z) - f(x)}{z-x} = \frac{-2}{(z+3)(x+3)} \rightarrow -\frac{2}{(x+3)^2} \text{ as } z \rightarrow x$ $f'(x) = -\frac{2}{(x+3)^2}, \quad x \neq -3$	M1 common denominator A1 numerator $2(x-z)$ M1 divides by $(z-x)$, cancels A1 $f'(x) = -2/(x+3)^2$
(b)	$f(-1) = 1, \quad f'(-1) = -\frac{2}{4} = -\frac{1}{2}$ $y - 1 = -\frac{1}{2}(x + 1) \Rightarrow y = -\frac{x}{2} + \frac{1}{2} \quad (x + 2y = 1)$	B1 point $(-1, 1)$ and slope $-1/2$ A1 equation



Question 7 One-sided derivatives at $x = 1$		[4]
(a)	$f(1) = 3(1) - 2 = 1, \quad \lim_{x \rightarrow 1^-} x^2 = 1, \quad \lim_{x \rightarrow 1^+} (3x - 2) = 1$ All three agree, so f is continuous at $x = 1$.	B1
(b)	$f'_+(1) = \lim_{h \rightarrow 0^+} \frac{3(1+h) - 2 - 1}{h} = \lim_{h \rightarrow 0^+} \frac{3h}{h} = 3$ $f'_-(1) = \lim_{h \rightarrow 0^-} \frac{(1+h)^2 - 1}{h} = \lim_{h \rightarrow 0^-} (2+h) = 2$	B1 right-hand derivative 3 B1 left-hand derivative 2
(c)	The one-sided derivatives (3 and 2) are not equal, so f is not differentiable at $x = 1$. The graph is continuous but has a corner there.	B1 conclusion with reason

Question 8 Constants a and b for differentiability		[4]
	$4a + 2b = 4 \Rightarrow 2a + b = 2 \quad \dots(1) \quad (\text{continuity at } x = 2)$ $4a + b = 3 \quad \dots(2) \quad (\text{derivatives match at } x = 2: 2ax + b = 3)$ $(2) - (1): 2a = 1 \Rightarrow a = \frac{1}{2}, \quad b = 2 - 2a = 1$ Check: left derivative at $2 = 4(1/2) + 1 = 3 = \text{right derivative}$.	B1 continuity equation B1 derivative equation M1 solves the pair A1 $a = 1/2, b = 1$

Question 9 Segments graph of f and its derivative		[6]
(a)	f is not defined at $x = -2, 1, 3, 5$. At each of these the graph has a corner: the slopes of the two adjoining segments differ (2 and 0; 0 and $-3/2$; $-3/2$ and 0; 0 and 2), so there is no single tangent line.	A2 all four points (A1 for two or three) B1 reason: corners / unequal one-sided slopes
(b)	$f'(x) = 2 \quad (-4 < x < -2), \quad 0 \quad (-2 < x < 1), \quad -\frac{3}{2} \quad (1 < x < 3),$ $0 \quad (3 < x < 5), \quad 2 \quad (5 < x < 6)$ 	B2 correct horizontal steps (B1 if three or four correct) B1 open circles / breaks at $x = -2, 1, 3, 5$



SECTION C: Differentiation Rules

[20 marks]

Question 10 First and second derivatives [4]		
(a)	$y = \frac{1}{2}x^4 - 4x^2 + 3x^{-1} + 9$ $\frac{dy}{dx} = 2x^3 - 8x - \frac{3}{x^2}$	B1 $2x^3 - 8x$ B1 $-3/x^2$
(b)	$\frac{d^2y}{dx^2} = 6x^2 - 8 + \frac{6}{x^3}$	B1 $6x^2 - 8$ B1 $6/x^3$
Question 11 Quotient and Product Rules [4]		
(a)	$y' = \frac{3(x^2 + 1) - (3x - 2)(2x)}{(x^2 + 1)^2} = \frac{-3x^2 + 4x + 3}{(x^2 + 1)^2}$	M1 correct quotient rule A1 simplified
(b)	$f'(t) = 2t\left(t - \frac{1}{t}\right) + (t^2 + 3)\left(1 + \frac{1}{t^2}\right) = 3t^2 + 2 + \frac{3}{t^2}$ Check by expanding first: $f(t) = t^3 + 2t - 3/t$, so $f'(t) = 3t^2 + 2 + 3/t^2$.	M1 correct product rule A1 simplified
Question 12 Derivatives from given values [3]		
(a)	$(uv)' = u'v + uv' = (-2)(-1) + (3)(4) = 14$	B1
(b)	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = \frac{(-2)(-1) - (3)(4)}{(-1)^2} = -10$	B1
(c)	$(3v - 2u)' = 3v' - 2u' = 3(4) - 2(-2) = 16$	B1
Question 13 $y = x^3 - 3x + 2$ [5]		
(a)	$y' = 3x^2 - 3, \quad y'(2) = 9$ Tangent: $y - 4 = 9(x - 2) \Rightarrow y = 9x - 14$ Normal: $y - 4 = -\frac{1}{9}(x - 2) \Rightarrow x + 9y = 38 \quad \left(y = -\frac{x}{9} + \frac{38}{9}\right)$	B1 slope 9 B1 tangent B1 normal (slope -1/9)
(b)	$3x^2 - 3 = 0 \Rightarrow x = \pm 1$ Points: (1, 0) and (-1, 4) .	M1 sets $y' = 0$ and solves A1 both points



Question 14	Tangents to $y = x^2$ through (1, -3)	[4]
	Point of tangency (a, a^2), slope $2a$: $y - a^2 = 2a(x - a)$ Through (1, -3): $-3 - a^2 = 2a(1 - a) \Rightarrow a^2 - 2a - 3 = 0$ $(a - 3)(a + 1) = 0 \Rightarrow a = 3$ or $a = -1$ $a = 3$: point (3, 9), tangent $y = 6x - 9$ $a = -1$: point (-1, 1), tangent $y = -2x - 1$	B1 slope $2a$ and tangent equation M1 substitutes (1, -3), forms quadratic A1 $a = 3$ and $a = -1$ A1 both points and both tangent lines

End of marking scheme. Dev's Educare: Soaring to Excellence.